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The gradient allocation principle based on the higher moment risk measure

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ABSTRACT

According to the gradient allocation principle based on a positively homogeneous and subadditive risk measure, the capital allocated to a sub-portfolio is the Gâteaux derivative, assuming it exists, of the underlying risk measure at the overall portfolio in the direction of the sub-portfolio. We consider the capital allocation problem based on the higher moment risk measure, which, as a generalization of expected shortfall, involves a risk aversion parameter and a confidence level and is consistent with the stochastic dominance of corresponding orders. As the main contribution, we prove that the higher moment risk measure is Gâteaux differentiable and derive an explicit expression for the Gâteaux derivative, which is then interpreted as the capital allocated to a corresponding sub-portfolio. We further establish the almost sure convergence and a central limit theorem for the empirical estimate of the capital allocation, and address the robustness issue of this empirical estimate by computing the influence function of the capital allocation. We also explore the interplay of the risk aversion and the confidence level in the context of capital allocation. In addition, we conduct intensive numerical studies to examine the obtained results and apply this research to a hypothetical portfolio of four stocks based on real data.

1. Introduction

Consider a financial firm whose business is composed of d different sub-portfolios. We use d real-valued random variables $X_1,\ldots, X_d$ to represent their loss-profit variables, simply called losses, during one time period. Accordingly, the firm faces a portfolio loss

$$S=\sum_{i=1}^{d} X_i. \tag{1.1}$$

The firm must set aside a certain amount of risk capital for buffering the firm's portfolio loss. The purpose of the backing is to protect stakeholders from potential insolvency of the firm in adverse situations. While it is important to determine the risk capital requirement for the portfolio loss S, it is of the same importance to allocate it amongst the sub-portfolios $X_1,\ldots, X_d$. A two-step procedure is often employed for capital allocation. The first step is to compute $\rho(S)$, for a given risk measure ρ, to determine the risk-based capital requirement. The second step is to allocate the risk capital $\rho(S)$ to the sub-portfolios conforming to a certain mathematical capital allocation rule Λ, such that the amounts $\Lambda(X_i,S)$ allocated to the individual sub-portfolios $i=1,\ldots,d$ sum to $\rho(S)$, namely,

$$\rho(S)=\sum_{i=1}^{d}\Lambda(X_i,S).$$

Thus, the risk manager needs to choose an appropriate risk measure ρ that determines the portfolio risk capital $\rho(S)$ and to choose a capital allocation rule Λ to efficiently and wisely allocate the risk capital to achieve the highest return on the overall portfolio.

Merton and Perold (1993) provide an excellent explanation of the need for prudent allocation of risk capital: "*In general, the incremental risk capital of a particular business within the firm will differ from its risk capital determined on the basis of a stand-alone analysis. As we shall demonstrate, this results from a diversification effect that can dramatically reduce the firm's overall risk capital.*" They con-

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clude: "*Full allocation of risk capital across the individual businesses of the firm therefore is generally not feasible. Attempts at such a full allocation can significantly distort the true profitability of individual businesses.*"

Capital allocation has been greatly developed in the past two decades and become an important approach to portfolio management. The literature on capital allocation is extensive, where a variety of principles for implementing the described two-step procedure are available.

Capital allocation originates in time-honored game theory in the form of cost allocation. Many works in capital allocation mainly employ an allocation rule that is equivalent to using the Aumann–Shapley value in game theory. The reader is referred to Tijs and Driessen (1986) for an early work on the connection from game theoretical concepts to cost allocation strategies, and to Powers (2007), Tsanakas (2009), and Boonen et al. (2020), among many others, for recent developments.

Denault (2001) is devoted to an axiomatic description of capital allocation by identifying conditions corresponding to those in game theory such that the Aumann–Shapley value gives a coherent allocation. This yields the gradient allocation principle, often also called the Euler principle, which has now become one of the most important allocation principles. Tasche (2004) provides a well-known justification that, as restated in Section 8.5.3 of McNeil et al. (2015), for a positively homogeneous risk measure, the gradient allocation principle is the only principle that is compatible with the performance measurement by the return on risk-adjusted capital (RORAC).

Kalkbrener (2005) gives a rigorous theoretical treatment of the gradient allocation principle. The author proposes an axiomatic system for capital allocation composed of linear aggregation, diversification, and continuity, and shows that, for a given risk measure ρ, there exists a capital allocation Λ that satisfies these axioms if and only if ρ is positively homogeneous and subadditive. Furthermore, if ρ is Gâteaux differentiable at the portfolio S in the sense that all directional derivatives of ρ at S exist, then the corresponding capital allocation principle Λ reduces to the gradient allocation principle under which the capital allocated to each sub-portfolio i is given by the Gâteaux derivative

$$\Lambda(X_i, S) = \lim_{\varepsilon \to 0} \frac{\rho(S + \varepsilon X_i) - \rho(S)}{\varepsilon}. \tag{1.2}$$

Proposition 3.1 of Fischer (2003) and Theorem 4.3 of Kalkbrener (2005) show equivalent conditions for the Gâteaux differentiability of a positively homogeneous and subadditive risk measure ρ.[1] Section 8.5.2 of McNeil et al. (2015) collects a number of special cases where the Gâteaux derivative (1.2) can be calculated explicitly. These examples allow the risk measure ρ to be the standard deviation, value at risk (VaR), and expected shortfall (ES), or allow the distribution of the vector $\mathbf{X} = (X_1, \ldots, X_d)^\top$ to be elliptical. For general cases, however, it is often very cumbersome or even impossible to explicitly calculate the Gâteaux derivative.

As other subjects in risk management, capital allocation has received some critical comments. In a series of papers, Phillips et al. (1998), Myers and Read (2001), and Gründl and Schmeiser (2007) dispute, from the angle of insurance pricing, whether or not the surplus of an insurance company should be allocated across the company's different lines of business. Later, Erel et al. (2015) develop a theoretical framework to show that, in the presence of tax and other costs, risk capital should be allocated. Buch et al. (2011), through their study of RORAC optimization, show that the implementation of the gradient capital allocation can be suboptimal for a decentralized financial firm in which division managers are allowed to venture into all business with higher marginal RORAC. For further discussions of issues with capital allocation, the reader is referred to Bauer and Zanjani (2013, 2016) and Chong et al. (2021), among others.

In capital allocation, there is no agreement on which risk measure ρ or which capital allocation rule Λ to choose. Instead, there is always a trade-off issue between practical usefulness and theoretical viability. On the one hand, ρ should reflect the firm's attitude towards the overall risk of the portfolio and Λ should reflect how the firm perceives the marginal contributions of the sub-portfolios to the overall risk, while on the other hand, both ρ and Λ should be well justified and possess some appealing properties. In this paper, we will adopt the axiomatic system of Kalkbrener (2005) so that the corresponding capital allocation rule Λ will be induced by a given risk measure ρ that is positively homogeneous, subadditive, and Gâteaux differentiable.[2]

Talking in the broader literature of risk management, it is always a controversial issue which risk measure is the best. The issue needs to be considered in the given economic context; see Bauer and Zanjani (2016), Emmer et al. (2015), Righi and Borenstein (2018), and He et al. (2022) for related discussions. Focused on different aspects of a risk variable, a large number of risk measures have been proposed, roughly categorized as monetary/loss risk measures and variability risk measures. ES, together with its various variants, has been in the spotlight of risk management for over two decades. Note that conditional value at risk (CVaR) and average value at risk (AVaR), which are originally proposed from different perspectives, are identical to ES. Earlier works on this strand of research include Rockafellar and Uryasev (2000, 2002), Acerbi and Tasche (2002), and Tasche (2002). The Basel Committee on Banking Supervision (BCBS) Working Paper (2011) points out that ES, thought of as a special spectral measure, is bound to a single confidence level and does not reflect the risk aversion varying with the loss size.[3] Koch-Medina and Munari (2016) point out several issues with ES-based regulation and highlight the need for its cautious use in capital adequacy regimes and portfolio risk control. Furman et al. (2017) further point out that ES does not capture the variability of a risk in the tail area and, motivated by this, they introduce Gini-type measures of risk and variability. Most recently, Burzoni et al. (2022) make a new effort to refine ES in view of its failure to distinguish across different tail behaviors with the same mean.

In this paper, we consider the capital allocation problem based on the higher moment (HM) risk measure $\rho_{p,q}(S)$, which involves a risk aversion parameter $p \geq 1$ and a confidence level $0 < q < 1$. Though first introduced by Krokhmal (2007) in the context of stochastic optimization, the HM risk measure has its root in risk management. When $p = 1$, it retrieves the ES risk measure. Importantly, in Lemma 2.2 we show that the HM risk measure $\rho_{p,q}$ is consistent with the classical stochastic dominance (SD) of order $p+1$. Thus, the HM risk measure is akin to the upper partial moment risk measure, the latter of which is mentioned in the BCBS Working Paper (2011) and Consultative Document (2012)

[1] Note that there are cases where the underlying risk measure ρ is not homogeneous but the Gâteaux derivative (1.2) still exists. Tsanakas (2009) discusses such cases and introduces the Aumann–Shapley capital allocation rule, to distinguish it from the capital allocation rule in the sense of Kalkbrener (2005).

[2] We note that Bauer and Zanjani (2016) reverse the usual approach in this literature by proposing to first calculate the marginal cost based on economic fundamentals and then identify a risk measure delivering the correct marginal cost. They use two examples to compare their results with those from the traditional approach based on VaR and ES and find that the VaR and ES capital allocations generally fail to weigh default outcomes properly. More recently, Chong et al. (2021) challenge the conventional two-step procedure by pointing out its three pitfalls and they instead introduce a holistic approach to capital allocation.

[3] See page 23 of the BCBS Working Paper (2011) "Messages from the academic literature on risk measurement for the trading book" available at https://www.bis.org/publ/bcbs_wp19.pdf.